

# Recurrence Relations and Fast Algorithms

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## Abstract

We construct fast algorithms for evaluating transforms associated with families of functions which satisfy recurrence relations. These include algorithms both for computing the coefficients in linear combinations of the functions, given the values of these linear combinations at certain points, and, vice versa, for evaluating such linear combinations at those points, given the coefficients in the linear combinations; such procedures are also known as analysis and synthesis of series of certain special functions. The algorithms of the present paper are efficient in the sense that their computational costs are proportional to  $n (\ln n) (\ln(1/\varepsilon))^3$ , where  $n$  is the amount of input and output data, and  $\varepsilon$  is the precision of computations. Stated somewhat more precisely, we find a positive real number  $C$  such that, for any positive integer  $n \geq 10$  and positive real number  $\varepsilon \leq 1/10$ , the algorithms require at most  $C n (\ln n) (\ln(1/\varepsilon))^3$  floating-point operations and words of memory to evaluate at  $n$  appropriately chosen points any linear combination of  $n$  special functions, given the coefficients in the linear combination, where  $\varepsilon$  is the precision of computations.

## 1 Introduction

Over the past several decades, the Fast Fourier Transform (FFT) and its variants (see, for example, [11]) have had an enormous impact across the sciences. The FFT is an efficient algorithm for computing, for any positive integer  $n$  and complex numbers  $\beta_1, \beta_2, \dots, \beta_{n-1}, \beta_n$ , the complex numbers  $\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n$  defined by

$$\alpha_j = \sum_{k=1}^n \beta_k f_k(x_j) \quad (1)$$

for  $j = 1, 2, \dots, n-1, n$ , where  $f_1, f_2, \dots, f_{n-1}, f_n$  are the functions defined on  $[-1, 1]$  by

$$f_k(x) = \exp\left(\frac{\pi i (2k - n) x}{2}\right) \quad (2)$$

for  $k = 1, 2, \dots, n-1, n$ , and  $x_1, x_2, \dots, x_{n-1}, x_n$  are the real numbers defined by

$$x_k = \frac{2k - n}{n} \quad (3)$$

for  $k = 1, 2, \dots, n-1, n$ . The FFT is efficient in the sense that there exists a reasonably small positive real number  $C$  such that, for any positive integer  $n \geq 10$ , the FFT requires at most  $C n \ln n$  floating-point operations and words of memory to compute  $\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n$  in (1) from  $\beta_1, \beta_2, \dots, \beta_{n-1}, \beta_n$ . In contrast, evaluating the sum in (1) separately for every  $j = 1, 2, \dots, n-1, n$  costs at least  $n^2$  operations in total.

The present paper introduces similarly efficient algorithms for computing  $\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n$  in (1) from  $\beta_1, \beta_2, \dots, \beta_{n-1}, \beta_n$ , and (when appropriate) for the inverse procedure of computing  $\beta_1, \beta_2, \dots, \beta_{n-1}, \beta_n$  from  $\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n$ , for more general collections of functions  $f_1, f_2, \dots, f_{n-1}, f_n$  and points  $x_1, x_2, \dots, x_{n-1}, x_n$  than those defined in (2) and (3). Specifically, the present paper constructs algorithms for classes of functions satisfying recurrence relations. The present paper describes in detail a few representative examples of such classes of functions, namely weighted orthonormal polynomials and Bessel functions of varying orders. These collections of functions satisfy recurrence relations of the form

$$g(x) f_k(x) = c_{k-1} f_{k-1}(x) + d_k f_k(x) + c_k f_{k+1}(x) \quad (4)$$

for all  $x$  in the domain, where  $c_{k-1}, c_k$ , and  $d_k$  are real numbers and either  $g(x) = x$  or  $g(x) = \frac{1}{x}$ ;  $c_k, d_k$ , and  $g$  vary with the collection of functions under consideration.

The algorithms of the present paper all rely on the following two observations:

1. The solutions to the recurrence relation (4) are the eigenvectors corresponding to eigenvalues  $g(x)$  of certain tridiagonal real self-adjoint matrices.
2. There exist fast algorithms for determining and applying matrices whose columns are normalized eigenvectors of a tridiagonal real self-adjoint matrix, and for applying the adjoints of these matrices of eigenvectors.

The first observation has been well known to numerical analysts at least since the seminal [3] appeared; the second observation has been reasonably well known to numerical analysts since the appearance of the celebrated [5]. However, the combination seems to be new.

The methods described in the present paper should lead to fairly efficient codes for computing a variety of what are known as (pseudo)spectral transforms. In particular, we can use the methods to construct fast algorithms for calculations involving spherical harmonics (see Remark 37 below).

We refer the reader to [13] and its compilation of references for prior work on related fast algorithms, as well as to [7] for an alternative approach that is suitable for certain applications, and to [9] for its refined accounting of computational costs. The present paper introduces techniques that are substantially more efficient than the extremely similar ones for which [13] reports on far-from-optimal implementations. We intend to report separately on carefully optimized implementations of the techniques described in the present paper, based in part upon the approach introduced in [8]. We gave a preliminary version of the present paper in [15].

The present paper has the following structure: Subsection 2.1 summarizes properties of fast algorithms for spectral representations of tridiagonal real self-adjoint matrices, Subsection 2.2 reiterates facts having to do with recurrence relations for orthonormal polynomials, Subsection 2.3 reiterates facts having to do with recurrence relations for Bessel functions, and Section 3 employs the subsections of Section 2 to construct fast algorithms for various purposes.

## 2 Preliminaries

This section summarizes certain widely known facts from numerical and mathematical analysis, used in Section 3.

### 2.1 Divide-and-conquer spectral methods

This subsection summarizes properties of fast algorithms introduced in [4] and [5] for spectral representations of tridiagonal real self-adjoint matrices. Specifically, there exists an algorithm such that, for any tridiagonal real self-adjoint matrix  $T$ , (firstly) the algorithm computes the eigenvalues of  $T$ , (secondly) the algorithm computes any eigenvector of  $T$ , (thirdly) the algorithm applies a square matrix  $U$  consisting of normalized eigenvectors of  $T$  to any arbitrary column vector, and (fourthly) the algorithm applies  $U^T$  to any arbitrary column vector, all using a number of operations and words of memory proportional to  $n (\ln n) (\ln(1/\varepsilon))^3$ , where  $n$  is the positive

integer for which  $T$  and  $U$  are  $n \times n$ , and  $\varepsilon$  is the precision of computations. The following is a more precise formulation.

For any positive integer  $n$ , self-adjoint  $n \times n$  matrix  $T$ , and real  $n \times 1$  column vector  $v$ , we define  $\|T\|$  to be the largest of the absolute values of the eigenvalues of  $T$ ,  $\delta_T$  to be the minimum value of the distance  $|\lambda - \mu|$  between any two distinct eigenvalues  $\lambda$  and  $\mu$  of  $T$ , and

$$\|v\| = \sqrt{\sum_{k=1}^n (v_k)^2}, \quad (5)$$

where  $v_1, v_2, \dots, v_{n-1}, v_n$  are the entries of  $v$ ; we say that  $v$  is *normalized* to mean that  $\|v\| = 1$ . As originated in [5], there exist an algorithm and a positive real number  $C$  such that, for any positive real number  $\varepsilon \leq 1/10$ , positive integer  $n \geq 10$ , tridiagonal real self-adjoint  $n \times n$  matrix  $T$  with  $n$  distinct eigenvalues, real unitary matrix  $U$  whose columns are  $n$  normalized eigenvectors of  $T$ , and real  $n \times 1$  column vector  $v$ ,

1. the algorithm computes to absolute precision  $\|T\| \varepsilon$  the  $n$  eigenvalues of  $T$ , using at most

$$C n (\ln n) (\ln(1/\varepsilon))^3 \quad (6)$$

floating-point operations and words of memory,

2. the algorithm computes to absolute precision  $\|T\| \|v\| \varepsilon / \delta_T$  the  $n$  entries of the matrix-vector product  $U v$ , using at most

$$C n (\ln n) (\ln(1/\varepsilon))^3 \quad (7)$$

operations and words of memory,

3. the algorithm computes to absolute precision  $\|T\| \|v\| \varepsilon / \delta_T$  the  $n$  entries of the matrix-vector product  $U^T v$ , using at most

$$C n (\ln n) (\ln(1/\varepsilon))^3 \quad (8)$$

operations and words of memory, and,

4. after the algorithm performs some precomputations which are particular to  $T$  at a cost of at most

$$C n (\ln n) (\ln(1/\varepsilon))^3 \quad (9)$$

operations and words of memory, the algorithm computes to absolute precision  $\|T\| \varepsilon / \delta_T$  the  $k n$  entries of any  $k$  normalized eigenvectors of  $T$ , using at most

$$C k n (\ln(1/\varepsilon))^2 \quad (10)$$

operations and words of memory, for any positive integer  $k$ .

**Remark 1** We omitted distracting factors of very small powers of  $n$  in the precisions mentioned in the present subsection. Also, the bounds on the number of operations and words of memory are extremely conservative; in actual implementations the running-times of the algorithm appear to scale much better with respect to the precision  $\varepsilon$ .

**Remark 2** In the second item of the present subsection, the algorithm in fact requires at most

$$C k n (\ln n) (\ln(1/\varepsilon))^2 \quad (11)$$

operations and words of memory to compute the matrix-vector products  $U v^1, U v^2, \dots, U v^{k-1}, U v^k$ , for any positive integer  $k$ , and real  $n \times 1$  column vectors  $v^1, v^2, \dots, v^{k-1}, v^k$ , after the algorithm performs some precomputations which are particular to  $T$  at a cost of at most

$$C n (\ln n) (\ln(1/\varepsilon))^3 \quad (12)$$

operations and words of memory. Moreover, we can improve the precisions to which the algorithm calculates  $U v^1, U v^2, \dots, U v^{k-1}, U v^k$ , by performing more expensive precomputations (using higher-precision floating-point arithmetic or precomputation algorithms whose costs are not proportional to  $n \ln n$ , for example). Similar considerations apply to the third item of the present subsection.

**Remark 3** There exist similar algorithms when the eigenvalues of  $T$  are not all distinct.

## 2.2 Orthonormal polynomials

This subsection discusses several classical facts concerning orthonormal polynomials. All of these facts follow trivially from results contained, for example, in [14].

Lemmas 8, 9, and 10, which formulate certain simple consequences of Theorems 4 and 7, are the principal tools used in Subsections 3.1 and 3.3. Lemmas 6 and 17 provide the results of some calculations for what are known as normalized Jacobi polynomials, a classical example of a family of orthonormal polynomials; the results of analogous calculations for some other classical families of polynomials are similar and therefore have been omitted. The remaining lemmas in the present subsection, Lemmas 12 and 15, deal with certain conditioning issues surrounding the algorithms in Subsections 3.1 and 3.3 (see Remark 16). The remaining theorem in the present subsection, Theorem 14, describes what are known as Gauss-Jacobi quadrature formulae.

In the present subsection, we index vectors and matrices starting at entry 0.

We say that  $a$  is an *extended real number* to mean that  $a$  is a real number,  $a = +\infty$ , or  $a = -\infty$ . For any real number  $a$ , we define the intervals  $[a, \infty] = [a, \infty)$  and  $[-\infty, a] = (-\infty, a]$ ; we define  $[-\infty, \infty] = (-\infty, \infty)$ .

For any extended real numbers  $a$  and  $b$  with  $a < b$  and nonnegative integer  $n$ , we say that  $p_0, p_1, \dots, p_{n-1}, p_n$  are *orthonormal polynomials on  $[a, b]$  for a weight  $w$*  to mean that  $w$  is a real-valued nonnegative integrable function on  $[a, b]$ ,  $p_k$  is a polynomial of degree  $k$ , the coefficients of  $x^0, x^1, \dots, x^{k-1}$ ,  $x^k$  in  $p_k(x)$  are real, and the coefficient of  $x^k$  in  $p_k(x)$  is positive for  $k = 0, 1, \dots, n-1, n$ , and

$$\int_a^b dx w(x) p_j(x) p_k(x) = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \quad (13)$$

for  $j, k = 0, 1, \dots, n-1, n$ .

The following theorem states that a system of orthonormal polynomials satisfies a certain three-term recurrence relation.

**Theorem 4** Suppose that  $a$  and  $b$  are extended real numbers with  $a < b$ ,  $n$  is a positive integer, and  $p_0, p_1, \dots, p_{n-1}, p_n$  are orthonormal polynomials on  $[a, b]$ .

Then, there exist real numbers  $c_0, c_1, \dots, c_{n-2}, c_{n-1}$  and  $d_0, d_1, \dots, d_{n-2}, d_{n-1}$  such that

$$x p_0(x) = d_0 p_0(x) + c_0 p_1(x) \quad (14)$$

for any  $x \in [a, b]$ , and

$$x p_k(x) = c_{k-1} p_{k-1}(x) + d_k p_k(x) + c_k p_{k+1}(x) \quad (15)$$

for any  $x \in [a, b]$  and  $k = 1, 2, \dots, n-2, n-1$ .

**Proof.** Theorem 3.2.1 in [14] provides an equivalent formulation of the present theorem.  $\square$

**Remark 5** In fact,  $c_k > 0$  for  $k = 0, 1, \dots, n-2, n-1$ , in (14) and (15).

The following lemma provides expressions for  $c_0, c_1, \dots, c_{n-2}, c_{n-1}$  and  $d_0, d_1, \dots, d_{n-2}, d_{n-1}$  from (14) and (15) for what are known as normalized Jacobi polynomials.

**Lemma 6** Suppose that  $a = -1$ ,  $b = 1$ ,  $\alpha$  and  $\beta$  are real numbers with  $\alpha > -1$  and  $\beta > -1$ ,  $n$  is a positive integer, and  $p_0, p_1, \dots, p_{n-1}, p_n$  are the orthonormal polynomials on  $[a, b]$  for the weight  $w$  defined by

$$w(x) = (1-x)^\alpha (1+x)^\beta. \quad (16)$$

Then,

$$c_k = \sqrt{\frac{4(k+1)(k+\alpha+1)(k+\beta+1)(k+\alpha+\beta+1)}{(2k+\alpha+\beta+1)(2k+\alpha+\beta+2)^2(2k+\alpha+\beta+3)}} \quad (17)$$

and

$$d_k = \frac{\beta^2 - \alpha^2}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)} \quad (18)$$

for  $k = 0, 1, \dots, n-2, n-1$ , where  $c_0, c_1, \dots, c_{n-2}, c_{n-1}$  and  $d_0, d_1, \dots, d_{n-2}, d_{n-1}$  are from (14) and (15).

**Proof.** Formulae 4.5.1 and 4.3.4 in [14] together provide an equivalent formulation of the present lemma.  $\square$

The following theorem states that the polynomial of degree  $n$  in a system of orthonormal polynomials on  $[a, b]$  has  $n$  distinct zeros in  $[a, b]$ .

**Theorem 7** Suppose that  $a$  and  $b$  are extended real numbers with  $a < b$ ,  $n$  is a positive integer, and  $p_0, p_1, \dots, p_{n-1}, p_n$  are orthonormal polynomials on  $[a, b]$ .

Then, there exist distinct real numbers  $x_0, x_1, \dots, x_{n-2}, x_{n-1}$  such that  $x_k \in [a, b]$  and

$$p_n(x_k) = 0 \quad (19)$$

for  $k = 0, 1, \dots, n-2, n-1$ , and

$$x_j \neq x_k \quad (20)$$

when  $j \neq k$  for  $j, k = 0, 1, \dots, n-2, n-1$ .

**Proof.** Theorem 3.3.1 in [14] provides a slightly more general formulation of the present theorem.  $\square$

Suppose that  $a$  and  $b$  are extended real numbers with  $a < b$ ,  $n$  is a positive integer, and  $p_0, p_1, \dots, p_{n-1}, p_n$  are orthonormal polynomials on  $[a, b]$  for a weight  $w$ . We define  $T$  to be the tridiagonal real self-adjoint  $n \times n$  matrix with the entry

$$T_{j,k} = \begin{cases} c_{j-1}, & k = j-1 \\ d_j, & k = j \\ c_j, & k = j+1 \\ 0, & \text{otherwise (when } k < j-1 \text{ or } k > j+1) \end{cases} \quad (21)$$

for  $j, k = 0, 1, \dots, n-2, n-1$ , where  $c_0, c_1, \dots, c_{n-2}, c_{n-1}$  and  $d_0, d_1, \dots, d_{n-2}, d_{n-1}$  are from (14) and (15). For  $k = 0, 1, \dots, n-1, n$ , we define the function  $q_k$  on  $[a, b]$  by

$$q_k(x) = \sqrt{w(x)} p_k(x). \quad (22)$$

We define  $U$  to be the real  $n \times n$  matrix with the entry

$$U_{j,k} = \frac{q_j(x_k)}{\sqrt{\sum_{m=0}^{n-1} (q_m(x_k))^2}} \quad (23)$$

for  $j, k = 0, 1, \dots, n-2, n-1$ , where  $q_0, q_1, \dots, q_{n-2}, q_{n-1}$  are defined in (22), and  $x_0, x_1, \dots, x_{n-2}, x_{n-1}$  are from (19). We define  $\Lambda$  to be the diagonal real  $n \times n$  matrix with the entry

$$\Lambda_{j,k} = \begin{cases} x_j, & k = j \\ 0, & k \neq j \end{cases} \quad (24)$$

for  $j, k = 0, 1, \dots, n-2, n-1$ , where  $x_0, x_1, \dots, x_{n-2}, x_{n-1}$  are from (19). We define  $S$  to be the diagonal real  $n \times n$  matrix with the entry

$$S_{j,k} = \begin{cases} \sqrt{\sum_{m=0}^{n-1} (q_m(x_j))^2}, & k = j \\ 0, & k \neq j \end{cases} \quad (25)$$

for  $j, k = 0, 1, \dots, n-2, n-1$ , where  $q_0, q_1, \dots, q_{n-2}, q_{n-1}$  are defined in (22), and  $x_0, x_1, \dots, x_{n-2}, x_{n-1}$  are from (19). We define  $e$  to be the real  $n \times 1$  column vector with the entry

$$e_k = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \quad (26)$$

for  $k = 0, 1, \dots, n-2, n-1$ .

The following lemma states that  $U$  is a matrix of normalized eigenvectors of the tridiagonal real self-adjoint matrix  $T$ , and that  $\Lambda$  is a diagonal matrix whose diagonal entries are the eigenvalues of  $T$  (which, according to (20), are distinct).

**Lemma 8** Suppose that  $a$  and  $b$  are extended real numbers with  $a < b$ ,  $n$  is a positive integer, and  $p_0, p_1, \dots, p_{n-1}, p_n$  are orthonormal polynomials on  $[a, b]$  for a weight  $w$ .

Then,

$$U^T T U = \Lambda, \quad (27)$$

where  $T$  is defined in (21),  $U$  is defined in (23), and  $\Lambda$  is defined in (24). Moreover,  $U$  is real and unitary.



**Proof.** Combining (14), (15), and (19) yields that

$$T U = U \Lambda. \quad (28)$$

Combining (28), (23), (24), and (20) yields that  $U$  is a real matrix of normalized eigenvectors of  $T$ , with distinct corresponding eigenvalues. Therefore, since eigenvectors corresponding to distinct eigenvalues of a real self-adjoint matrix are orthogonal,  $U$  is orthogonal. Applying  $U^T$  from the left to both sides of (28) yields (27).  $\square$

The following lemma expresses in matrix notation the analysis and synthesis of linear combinations of weighted orthonormal polynomials for which Subsection 3.3 describes fast algorithms.

**Lemma 9** Suppose that  $a$  and  $b$  are extended real numbers with  $a < b$ ,  $n$  is a positive integer,  $p_0, p_1, \dots, p_{n-1}, p_n$  are orthonormal polynomials on  $[a, b]$  for a weight  $w$ , and  $\alpha$  and  $\beta$  are real  $n \times 1$  column vectors, such that  $\alpha$  has the entry

$$\alpha_j = \sum_{k=0}^{n-1} \beta_k q_k(x_j) \quad (29)$$

for  $j = 0, 1, \dots, n-2, n-1$ , where  $q_0, q_1, \dots, q_{n-2}, q_{n-1}$  are defined in (22), and  $x_0, x_1, \dots, x_{n-2}, x_{n-1}$  are from (19).

Then,

$$\alpha = S U^T \beta \quad (30)$$

and

$$\beta = U S^{-1} \alpha, \quad (31)$$

where  $U$  is defined in (23),  $S$  is defined in (25), and  $S U^T \beta$  and  $U S^{-1} \alpha$  are matrix-matrix-vector products.

**Proof.** Combining (23) and (25) yields (30). According to Lemma 8,  $U$  is real and unitary. Therefore, applying the matrix-matrix product  $U S^{-1}$  from the left to both sides of (30) yields (31).  $\square$

The following two lemmas provide alternative expressions for the entries of  $S$  defined in (25).

**Lemma 10** Suppose that  $a$  and  $b$  are extended real numbers with  $a < b$ ,  $n$  is a positive integer, and  $p_0, p_1, \dots, p_{n-1}, p_n$  are orthonormal polynomials on  $[a, b]$  for a weight  $w$ .

Then,

$$S_{k,k} = \frac{\sqrt{w(x_k)}}{(U^T e)_k \sqrt{\int_a^b dx w(x)}} \quad (32)$$

for  $k = 0, 1, \dots, n-2, n-1$ , where  $S$  is defined in (25),  $U$  is defined in (23),  $e$  is defined in (26),  $(U^T e)_0, (U^T e)_1, \dots, (U^T e)_{n-2}, (U^T e)_{n-1}$  are the entries of the matrix-vector product  $U^T e$ , and  $x_0, x_1, \dots, x_{n-2}, x_{n-1}$  are from (19).

**Proof.** Combining (23) and (26) yields that

$$(U^T e)_k = \frac{q_0(x_k)}{\sqrt{\sum_{m=0}^{n-1} (q_m(x_k))^2}} \quad (33)$$

for  $k = 0, 1, \dots, n-2, n-1$ . Since the polynomial  $p_0$  has degree 0, combining (13) and (22) yields that

$$q_0(x) = \frac{\sqrt{w(x)}}{\sqrt{\int_a^b dy w(y)}} \quad (34)$$

for any  $x \in [a, b]$ . Combining (25), (33), and (34) yields (32).  $\square$

**Remark 11** Formula 2.6 in [3] motivated us to employ the equivalent (32).

**Lemma 12** Suppose that  $a$  and  $b$  are extended real numbers with  $a < b$ ,  $n$  is a positive integer,  $p_0, p_1, \dots, p_{n-1}, p_n$  are orthonormal polynomials on  $[a, b]$  for a weight  $w$ , and  $k$  is a nonnegative integer, such that  $\ln w$  is differentiable at the point  $x_k$  from (19).

Then,

$$(S_{k,k})^2 = c_{n-1} q_{n-1}(x_k) \frac{d}{dx} q_n(x_k), \quad (35)$$

where  $S_{k,k}$  is defined in (25),  $c_{n-1}$  is from (15),  $q_{n-1}$  and  $q_n$  are defined in (22), and  $x_k$  is from (19).

**Proof.** Formula 3.2.4 in [14] provides a slightly more general formulation of the present lemma.  $\square$

**Remark 13** There exist similar formulations of Lemma 12 when it is not the case that  $\ln w$  is differentiable at  $x_k$ .

The following theorem describes what are known as Gauss-Jacobi quadrature formulae for orthonormal polynomials.

**Theorem 14** Suppose that  $a$  and  $b$  are extended real numbers with  $a < b$ ,  $n$  is a positive integer, and  $p_0, p_1, \dots, p_{n-1}, p_n$  are orthonormal polynomials on  $[a, b]$  for a weight  $w$ .

Then, there exist positive real numbers  $w_0, w_1, \dots, w_{n-2}, w_{n-1}$ , called the Christoffel numbers for  $x_0, x_1, \dots, x_{n-2}, x_{n-1}$ , such that

$$\int_a^b dx w(x) p(x) = \sum_{k=0}^{n-1} w_k p(x_k) \quad (36)$$

for any polynomial  $p$  of degree at most  $2n-1$ , where  $x_0, x_1, \dots, x_{n-2}, x_{n-1}$  are from (19).

**Proof.** Theorems 3.4.1 and 3.4.2 in [14] together provide a slightly more general formulation of the present theorem.  $\square$

The following lemma provides alternative expressions for the entries of  $S$  defined in (25).

**Lemma 15** Suppose that  $a$  and  $b$  are extended real numbers with  $a < b$ ,  $n$  is a positive integer, and  $p_0, p_1, \dots, p_{n-1}, p_n$  are orthonormal polynomials on  $[a, b]$  for a weight  $w$ .

Then,

$$(S_{k,k})^2 = \frac{w(x_k)}{w_k} \quad (37)$$

for  $k = 0, 1, \dots, n-2, n-1$ , where  $S$  is defined in (25), and  $w_0, w_1, \dots, w_{n-2}, w_{n-1}$  are the Christoffel numbers from (36) for the corresponding points  $x_0, x_1, \dots, x_{n-2}, x_{n-1}$  from (19). Moreover, there exist extended real numbers  $y_0, y_1, \dots, y_{n-1}, y_n$  such that  $a = y_0 < y_1 < \dots < y_{n-1} < y_n = b$  and

$$w_k = \int_{y_k}^{y_{k+1}} dx w(x) \quad (38)$$

for  $k = 0, 1, \dots, n-2, n-1$ .

**Proof.** Formula 3.4.8 in [14] provides an equivalent formulation of (37). Formula 3.41.1 in [14] provides a slightly more general formulation of (38).  $\square$

**Remark 16** The formulae (25), (35), (37), and (38) give some insight into the condition number of  $S$ . For instance, due to (25), the entries of  $S$  are usually not too large.

The following lemma provides an alternative expression for the entries of  $S$  defined in (25) for what are known as normalized Jacobi polynomials.

**Lemma 17** Suppose that  $a = -1$ ,  $b = 1$ ,  $\alpha$  and  $\beta$  are real numbers with  $\alpha > -1$  and  $\beta > -1$ ,  $n$  is a positive integer, and  $p_0, p_1, \dots, p_{n-1}, p_n$  are the orthonormal polynomials on  $[a, b]$  for the weight  $w$  defined by

$$w(x) = (1 - x)^\alpha (1 + x)^\beta. \quad (39)$$

Then,

$$S_{k,k} = \sqrt{\frac{1 - x_k^2}{2n + \alpha + \beta + 1}} \left| \frac{d}{dx} q_n(x_k) \right| \quad (40)$$

for  $k = 0, 1, \dots, n - 2, n - 1$ , where  $S$  is defined in (25),  $x_0, x_1, \dots, x_{n-2}, x_{n-1}$  are from (19), and  $q_n$  is defined in (22).

**Proof.** Together with (37), Formulae 15.3.1 and 4.3.4 in [14] provide an equivalent formulation of the present lemma.  $\square$

## 2.3 Bessel functions

This subsection discusses several well-known facts concerning Bessel functions. All of these facts follow trivially from results contained, for example, in [17] and [10].

Lemmas 25, 26, 27, and 28 are the principal tools used in Subsections 3.2 and 3.4. These lemmas formulate certain simple consequences of Theorems 19 and 22 and Corollary 23, by way of Lemmas 20 and 24. Lemmas 32 and 33 provide closed-form results of some calculations for what are known as spherical Bessel functions, a family of Bessel functions frequently encountered in applications. The remaining lemmas in the present subsection, Lemmas 34 and 35, provide closed-form results of similar calculations for Bessel functions of arbitrary nonnegative orders (however, the results in Lemmas 34 and 35 arise from identities that are presumably not quite as familiar).

In the present subsection, we index vectors and matrices starting at entry 1.

Suppose that  $\nu$  is a nonnegative real number. For any nonnegative integer  $k$ , we define the function  $f_k$  on  $(0, \infty)$  by

$$f_k(x) = \frac{2^\nu \Gamma(\nu + 1) \sqrt{\nu + k}}{x^\nu} J_{\nu+k}(x), \quad (41)$$

where  $\Gamma$  is the gamma (factorial) function and  $J_{\nu+k}$  is the Bessel function of the first kind of order  $\nu + k$  (see, for example, [17]).

**Remark 18** Formula 8 of Section 3.1 in [17] provides a more general formulation of the fact that

$$\lim_{x \rightarrow 0^+} \frac{2^\nu \Gamma(\nu + 1)}{x^\nu} J_\nu(x) = 1, \quad (42)$$

which motivated our choice of normalization in (41).

The following theorem states that  $f_1, f_2, f_3, \dots$  defined in (41) satisfy a certain three-term recurrence relation.

**Theorem 19** Suppose that  $\nu$  is a nonnegative real number.

Then,

$$\frac{1}{x} f_1(x) = \frac{1}{2\sqrt{(\nu+1)}} \frac{2^\nu \Gamma(\nu+1)}{x^\nu} J_\nu(x) + \frac{1}{2\sqrt{(\nu+1)(\nu+2)}} f_2(x) \quad (43)$$

for any positive real number  $x$ , and

$$\frac{1}{x} f_k(x) = \frac{1}{2\sqrt{(\nu+k-1)(\nu+k)}} f_{k-1}(x) + \frac{1}{2\sqrt{(\nu+k)(\nu+k+1)}} f_{k+1}(x) \quad (44)$$

for any positive real number  $x$  and  $k = 2, 3, 4, \dots$ , where  $f_1, f_2, f_3, \dots$  are defined in (41),  $\Gamma$  is the gamma (factorial) function, and  $J_\nu$  is the Bessel function of the first kind of order  $\nu$  (see, for example, [17]).

**Proof.** Formula 1 of Section 3.2 in [17] provides a somewhat more general formulation of the present theorem.  $\square$

Suppose that  $\nu$  is a nonnegative real number and  $n$  is a positive integer. We define  $T$  to be the tridiagonal real self-adjoint  $n \times n$  matrix with the entry

$$T_{j,k} = \begin{cases} \frac{1}{2\sqrt{(\nu+j-1)(\nu+j)}}, & k = j-1 \\ \frac{1}{2\sqrt{(\nu+j)(\nu+j+1)}}, & k = j+1 \\ 0, & \text{otherwise } (k < j-1, k = j, \text{ or } k > j+1) \end{cases} \quad (45)$$

for  $j, k = 1, 2, \dots, n-1, n$ . For any positive real number  $x$ , we define  $v = v(x)$  to be the real  $n \times 1$  column vector with the entry

$$v_k = \frac{f_k(x)}{\sqrt{\sum_{m=1}^n (f_m(x))^2}} \quad (46)$$

for  $k = 1, 2, \dots, n-1, n$ , where  $f_1, f_2, \dots, f_{n-1}, f_n$  are defined in (41). For any positive real number  $x$ , we define  $\delta = \delta(x)$  to be the real number

$$\delta = \frac{1}{2\sqrt{(\nu+n)(\nu+n+1)}} \frac{|f_{n+1}(x)|}{\sqrt{\sum_{m=1}^n (f_m(x))^2}}, \quad (47)$$

where  $f_1, f_2, \dots, f_n, f_{n+1}$  are defined in (41).

The following lemma states that  $v$  is nearly an eigenvector of the tridiagonal real self-adjoint matrix  $T$  corresponding to an approximate eigenvalue of  $\frac{1}{x}$  for any positive real number  $x$  such that  $J_\nu(x) = 0$  and  $\delta$  is small.

**Lemma 20** Suppose that  $\nu$  is a nonnegative real number and  $n$  is a positive integer.

Then,

$$\left| (Tv)_n - \frac{1}{x} v_n \right| \leq \delta \quad (48)$$

and

$$(Tv)_k = \frac{1}{x} v_k \quad (49)$$

for  $k = 1, 2, \dots, n-2, n-1$  and any positive real number  $x$  with

$$J_\nu(x) = 0, \quad (50)$$

where  $T$  is defined in (45),  $v = v(x)$  is defined in (46),  $(Tv)_1, (Tv)_2, \dots, (Tv)_{n-1}, (Tv)_n$  are the entries of the matrix-vector product  $Tv$ ,  $\delta = \delta(x)$  is defined in (47), and  $J_\nu$  is the Bessel function of the first kind of order  $\nu$  (see, for example, [17]).

**Proof.** Combining (44), (43), and (50) yields (48) and (49).  $\square$

**Remark 21** It is well known that, for any positive real number  $x$ , the quantity  $J_{\nu+n+1}(x)$  and thence  $\delta$  defined in (47) decays extremely rapidly as  $n$  increases past a band around  $n = x$  of width proportional to  $x^{1/3}$ ; see, for example, Lemma 2.5 in [12], Chapters 9 and 10 in [1], or Chapter 8 in [17]. Therefore,  $\delta$  is often small for  $x$  such that  $x < n$  and  $J_\nu(x) = 0$ .

The following theorem states a simple Sturm sequence property of the eigenvalues of real self-adjoint tridiagonal matrices whose entries on the sub- and super-diagonals are nonzero.

**Theorem 22** Suppose that  $n$  is a positive integer, and  $T$  is a tridiagonal real self-adjoint  $n \times n$  matrix, such that all entries on the sub- and super-diagonals of  $T$  are nonzero.

Then, every eigenvalue of  $T$  has multiplicity 1.

**Proof.** Formula 7-7-1 in [10] provides an equivalent formulation of the present theorem.  $\square$

The following corollary of Theorem 22 states a simple Sturm sequence property of the eigenvalues of  $T$  defined in (45).

**Corollary 23** Suppose that  $\nu$  is a nonnegative real number and  $n$  is a positive integer.

Then, every eigenvalue of  $T$  defined in (45) has multiplicity 1.

The following lemma bounds the distance between an approximate eigenvalue and the actual eigenvalue nearest to the approximation, as well as the discrepancy between the corresponding normalized approximate eigenvector and a corresponding normalized actual eigenvector.

**Lemma 24** Suppose that  $\gamma$ ,  $\lambda$ , and  $\mu$  are real numbers,  $n$  is a positive integer,  $T$  is a real self-adjoint  $n \times n$  matrix, and  $u$  and  $v$  are real  $n \times 1$  column vectors, such that  $\lambda$  is the eigenvalue of  $T$  nearest to  $\gamma$ ,  $\lambda$  has multiplicity 1,  $\mu$  is the eigenvalue of  $T$  nearest but not equal to  $\lambda$ ,

$$Tu = \lambda u, \quad (51)$$

$$\sum_{k=1}^n (u_k)^2 = 1, \quad (52)$$

and

$$\sum_{k=1}^n (v_k)^2 = 1. \quad (53)$$

Then,

$$|\gamma - \lambda| \leq \sqrt{\sum_{k=1}^n \left( (Tv)_k - \gamma v_k \right)^2}, \quad (54)$$

and either (or both)

$$|v_k - u_k| \leq \frac{2 \sqrt{\sum_{k=1}^n \left( (Tv)_k - \gamma v_k \right)^2}}{|\mu - \lambda|} \quad (55)$$

for  $k = 1, 2, \dots, n-1, n$ , or

$$|-v_k - u_k| \leq \frac{2 \sqrt{\sum_{k=1}^n \left( (Tv)_k - \gamma v_k \right)^2}}{|\mu - \lambda|} \quad (56)$$

for  $k = 1, 2, \dots, n-1, n$ , where  $(Tv)_1, (Tv)_2, \dots, (Tv)_{n-1}, (Tv)_n$  are the entries of the matrix-vector product  $Tv$ .

**Proof.** Formula 4-5-1 in [10] provides an equivalent formulation of (54).

The proof of Formula 11-7-1 in [10], specifically Formula 11-7-3 in [10] and the comment immediately following Formula 11-7-3 in [10], provides a slightly more general formulation of the fact that (55) holds for  $k = 1, 2, \dots, n-1, n$ , or that (56) holds for  $k = 1, 2, \dots, n-1, n$ .  $\square$

The following lemma bounds the changes in the eigenvalues and eigenvectors induced by using the truncated matrix  $T$  defined in (45) ( $T$  is only  $n \times n$ , not infinite-dimensional).

**Lemma 25** Suppose that  $\lambda, \mu, \nu$ , and  $x$  are real numbers,  $n$  is a positive integer, and  $u$  is a real  $n \times 1$  column vector, such that  $\nu \geq 0, x > 0$ , (50) holds,  $\lambda$  is the eigenvalue of  $T$  nearest to  $\frac{1}{x}$ ,  $\mu$  is the eigenvalue of  $T$  nearest but not equal to  $\lambda$ ,

$$T u = \lambda u, \quad (57)$$

and

$$\sum_{k=1}^n (u_k)^2 = 1, \quad (58)$$

where  $T$  is defined in (45).

Then,

$$\left| \frac{1}{x} - \lambda \right| \leq \delta, \quad (59)$$

and either (or both)

$$|v_k - u_k| \leq \frac{2\delta}{|\mu - \lambda|} \quad (60)$$

for  $k = 1, 2, \dots, n-1, n$ , or

$$|-v_k - u_k| \leq \frac{2\delta}{|\mu - \lambda|} \quad (61)$$

for  $k = 1, 2, \dots, n-1, n$ , where  $v = v(x)$  is defined in (46) and  $\delta = \delta(x)$  is defined in (47).

**Proof.** Combining Corollary 23, (54), (48), and (49) yields (59).

Combining Corollary 23, (55), (56), (48), and (49) yields that (60) holds for  $k = 1, 2, \dots, n-1, n$ , or that (61) holds for  $k = 1, 2, \dots, n-1, n$ .  $\square$

Suppose that  $\nu$  is a nonnegative real number and  $n$  is a positive integer. We define  $x_1, x_2, x_3, \dots$  to be all of the positive real numbers such that

$$J_\nu(x_k) = 0 \quad (62)$$



for any positive integer  $k$ , ordered so that

$$0 < x_1 < x_2 < x_3 < \dots, \quad (63)$$

where  $J_\nu$  is the Bessel function of the first kind of order  $\nu$  (see, for example, [17]). We define  $S$  to be the diagonal real  $n \times n$  matrix with the entry

$$S_{j,k} = \begin{cases} \sqrt{\sum_{m=1}^n (f_m(x_j))^2}, & j = k \\ 0, & j \neq k \end{cases} \quad (64)$$

for  $j, k = 1, 2, \dots, n-1, n$ , where  $f_1, f_2, \dots, f_{n-1}, f_n$  are defined in (41), and  $x_1, x_2, \dots, x_{n-1}, x_n$  are defined in (62) and (63). We define  $e$  to be the real  $n \times 1$  column vector with the entry

$$e_k = \begin{cases} 1, & k = 1 \\ 0, & k \neq 1 \end{cases} \quad (65)$$

for  $k = 1, 2, \dots, n-1, n$ .

The following lemma expresses in matrix notation the evaluations of linear combinations of Bessel functions for which Subsection 3.4 describes fast algorithms.

**Lemma 26** Suppose that  $\nu$  is a nonnegative real number,  $n$  is a positive integer, and  $\alpha$  and  $\beta$  are real  $n \times 1$  column vectors, such that  $\alpha$  has the entry

$$\alpha_j = \sum_{k=1}^n \beta_k f_k(x_j) \quad (66)$$

for  $j = 1, 2, \dots, n-1, n$ , where  $f_1, f_2, \dots, f_{n-1}, f_n$  are defined in (41), and  $x_1, x_2, \dots, x_{n-1}, x_n$  are defined in (62) and (63).

Then,

$$|\alpha_k - (S U^T \beta)_k| \leq \frac{2 S_{k,k} \delta(x_k)}{|\mu_k - \lambda_k|} \quad (67)$$

for any  $k = 1, 2, \dots, n-1, n$  such that

$$\frac{2 S_{k,k} \delta(x_k)}{|\mu_k - \lambda_k|} < |f_1(x_k)|, \quad (68)$$

where  $\lambda_k$  is the eigenvalue of  $T$  defined in (45) nearest to  $\frac{1}{x_k}$ ,  $\mu_k$  is the eigenvalue of  $T$  nearest but not equal to  $\lambda_k$ ,  $\delta = \delta(x_k)$  is defined in (47),  $U$  is a real  $n \times n$  matrix whose  $k^{\text{th}}$  column is the normalized eigenvector of  $T$  corresponding to the eigenvalue  $\lambda_k$  whose first entry has the same sign as  $f_1(x_k)$ ,  $S$  is defined in (64), and  $(S U^T \beta)_k$  is the  $k^{\text{th}}$  entry of the matrix-matrix-vector product  $S U^T \beta$ .

**Proof.** Combining (60), (46), and (64) yields (67).  $\square$

The following two lemmas provide alternative expressions for the entries of  $S$  defined in (64).

**Lemma 27** Suppose that  $\nu$  is a nonnegative real number and  $n$  is a positive integer.

Then,

$$\left| S_{k,k} - \frac{f_1(x_k)}{(U^T e)_k} \right| \leq \frac{2 S_{k,k} \delta(x_k)}{|\mu_k - \lambda_k| |(U^T e)_k|} \quad (69)$$

for any  $k = 1, 2, \dots, n-1, n$  such that (68) holds, where  $S$  is defined in (64),  $f_1$  is defined in (41),  $x_1, x_2, \dots, x_{n-1}, x_n$  are defined in (62) and (63),  $\lambda_k$  is the eigenvalue of  $T$  defined in (45) nearest to  $\frac{1}{x_k}$ ,  $\mu_k$  is the eigenvalue of  $T$  nearest but not equal to  $\lambda_k$ ,  $\delta = \delta(x_k)$  is defined in (47),  $U$  is a real  $n \times n$  matrix whose  $k^{\text{th}}$  column is the normalized eigenvector of  $T$  corresponding to the eigenvalue  $\lambda_k$  whose first entry has the same sign as  $f_1(x_k)$ ,  $e$  is defined in (65), and  $(U^T e)_k$  is the  $k^{\text{th}}$  entry of the matrix-vector product  $U^T e$ .

**Proof.** Combining (60), (46), and (65) yields that

$$\left| (U^T e)_k - \frac{f_1(x_k)}{\sqrt{\sum_{m=1}^n (f_m(x_k))^2}} \right| \leq \frac{2 \delta(x_k)}{|\mu_k - \lambda_k|} \quad (70)$$

for any  $k = 1, 2, \dots, n-1, n$  such that (68) holds. Combining (64) and (70) yields (69).  $\square$

**Lemma 28** Suppose that  $\nu$  is a nonnegative real number and  $n$  is a positive integer.

Then,

$$f_1(x_k) = -\frac{2^\nu \Gamma(\nu+1) \sqrt{\nu+1}}{(x_k)^\nu} \frac{d}{dx} J_\nu(x_k) \quad (71)$$

for  $k = 1, 2, \dots, n-1, n$ , where  $f_1$  is defined in (41),  $x_1, x_2, \dots, x_{n-1}, x_n$  are defined in (62) and (63),  $\Gamma$  is the gamma (factorial) function, and  $J_\nu$  is the Bessel function of the first kind of order  $\nu$  (see, for example, [17]).

**Proof.** Formula 4 of Section 3.2 in [17] provides a somewhat more general formulation of (71).  $\square$

**Remark 29** The right hand side of (69) involves the potentially troublesome

$$\frac{1}{|(U^T e)_k|}. \quad (72)$$

However, due to (60), (46), and (64), if

$$\frac{2 \delta(x_k)}{|\mu_k - \lambda_k|} \quad (73)$$

is small, then (72) is accordingly close to

$$\frac{S_{k,k}}{|f_1(x_k)|}, \quad (74)$$

which should not be unreasonably large.

**Remark 30** Numerical experiments indicate that  $|\mu_k - \lambda_k|$  in (67) and (69) is never exceedingly small for practical ranges of  $n$ ; this is probably fairly easy to prove, perhaps using the properties of Sturm sequences. The following remark appears to be relevant.

**Remark 31** Suppose that  $\nu = 0$  and  $n$  is a positive integer. For any positive real number  $x$ , we define  $\tilde{v} = \tilde{v}(x)$  to be the real  $n \times 1$  column vector with the entry

$$\tilde{v}_k = (-1)^k v_k \quad (75)$$

for  $k = 1, 2, \dots, n-1, n$ , where  $v = v(x)$  is defined in (46). Then, Formula 1 of Section 3.2 in [17] and Formula 2 of Section 2.1 in [17] lead to

$$\left| (T \tilde{v})_n + \frac{1}{x} \tilde{v}_n \right| \leq \delta \quad (76)$$

in place of (48),

$$(T \tilde{v})_k = -\frac{1}{x} \tilde{v}_k \quad (77)$$

for  $k = 1, 2, \dots, n-2, n-1$  in place of (49), etc.

The following lemma states a special case of the Gegenbauer addition formula for Bessel functions.

**Lemma 32** Suppose that  $\nu = \frac{1}{2}$ .

Then,

$$\sum_{m=0}^{\infty} (f_m(x))^2 = \frac{1}{2} \quad (78)$$

for any positive real number  $x$ , where  $f_0, f_1, f_2, \dots$  are defined in (41).

**Proof.** Formula 3 of Section 11.4 in [17] provides a somewhat more general formulation of (78).  $\square$

The following lemma provides alternative expressions for the entries of  $S$  defined in (64) for what are known as spherical Bessel functions of the first kind.

**Lemma 33** Suppose that  $\nu = \frac{1}{2}$ ,  $\varepsilon$  is a positive real number, and  $k$  and  $n$  are positive integers, such that

$$\sum_{m=n+1}^{\infty} (f_m(x_k))^2 \leq \varepsilon, \quad (79)$$

where  $f_{n+1}, f_{n+2}, f_{n+3}, \dots$  are defined in (41), and  $x_k$  is defined in (62) and (63).

Then,

$$\left| (S_{k,k})^2 - \frac{1}{2} \right| \leq \varepsilon, \quad (80)$$

where  $S_{k,k}$  is defined in (64).

**Proof.** Combining (64), (78), (79), and (62) yields (80).  $\square$

The following lemma provides a closed-form expression for the sum in (78), for any nonnegative order  $\nu$ .

**Lemma 34** Suppose that  $\nu$  is a nonnegative real number.

Then,

$$\begin{aligned} \sum_{m=1}^{\infty} (f_m(x))^2 &= \frac{x^2}{2(\nu+1)} (f_1(x))^2 + \frac{x^2}{2} \left( \frac{2^\nu \Gamma(\nu+1)}{x^\nu} \right)^2 (J_\nu(x))^2 \\ &\quad - \frac{(2\nu+1)x}{2\sqrt{\nu+1}} \left( \frac{2^\nu \Gamma(\nu+1)}{x^\nu} \right) J_\nu(x) f_1(x) \end{aligned} \quad (81)$$

for any positive real number  $x$ , where  $f_1, f_2, f_3, \dots$  are defined in (41),  $\Gamma$  is the gamma (factorial) function, and  $J_\nu$  is the Bessel function of the first kind of order  $\nu$  (see, for example, [17]).

**Proof.** Formula 57.21.1 of [6] provides a more general formulation of the present lemma. See also Formula 24 of [16] and the surrounding discussion for a self-contained derivation of the present lemma.  $\square$

The following lemma provides alternative expressions for the entries of  $S$  defined in (64).

**Lemma 35** Suppose that  $\nu$  and  $\varepsilon$  are real numbers, and  $k$  and  $n$  are positive integers, such that  $\nu \geq 0$ ,  $\varepsilon > 0$ , and (79) holds, where in (79),  $f_{n+1}, f_{n+2}, f_{n+3}, \dots$  are defined in (41), and  $x_k$  is defined in (62) and (63). Then,

$$\left| (S_{k,k})^2 - \frac{(x_k)^2}{2(\nu+1)} (f_1(x_k))^2 \right| \leq \varepsilon, \quad (82)$$

where  $S_{k,k}$  is defined in (64), and  $f_1$  is defined in (41).

**Proof.** Combining (64), (81), (79), and (62) yields (82).  $\square$

**Remark 36** As in Remark 21, it is often possible to have  $\varepsilon$  in (79), (80), and (82) be small for  $k$  such that  $x_k < n$ .

### 3 Fast algorithms

This section constructs efficient algorithms for computing the quadrature nodes and Christoffel numbers associated with orthonormal polynomials, for computing the zeros of Bessel functions, for the analysis and synthesis of linear combinations of weighted orthonormal polynomials, and for evaluations of linear combinations of Bessel functions. We describe the algorithms in Subsections 3.1 and 3.2 solely to illustrate the generality of the techniques discussed in the present paper; we would expect specialized schemes such as those in [2] to outperform the algorithms described in Subsections 3.1 and 3.2 in most, if not all, practical circumstances. Each subsection in the present section relies on both Subsection 2.1 and either Subsection 2.2 or Subsection 2.3.

#### 3.1 Quadrature nodes and Christoffel numbers associated with orthonormal polynomials

The entries of  $\Lambda$  in (27) are the nodes  $x_0, x_1, \dots, x_{n-2}, x_{n-1}$  in (36). We can compute rapidly the entries of  $\Lambda$  in (27) using an algorithm as in the first item in Subsection 2.1, due to (27), since  $T$  in (27) is tridiagonal, real, and self-adjoint,  $U$  in (27) is real and unitary, and  $\Lambda$  in (27) is diagonal, with diagonal entries that according to (20) are distinct. For the same reason, we can apply rapidly the matrix  $U^T$  to the vector  $e$  in (32) using an algorithm as in the third item in Subsection 2.1. We can then compute the Christoffel numbers  $w_0, w_1, \dots, w_{n-2}, w_{n-1}$  in (36) using (37) and (32).

### 3.2 Zeros of Bessel functions

We can compute rapidly the zeros  $x_1, x_2, \dots, x_{n-1}, x_n$  defined in (62) and (63) for which  $\delta$  defined in (47) is sufficiently small, using (59) and an algorithm as in the first item in Subsection 2.1, since  $T$  defined in (45) is tridiagonal, real, and self-adjoint, and (according to Corollary 23) has  $n$  distinct eigenvalues.

### 3.3 Analysis and synthesis of linear combinations of weighted orthonormal polynomials

We can apply rapidly the matrices  $U$  and  $U^T$  in (31) and (30) using an algorithm as in the second and third items in Subsection 2.1, due to (27), since  $T$  in (27) is tridiagonal, real, and self-adjoint,  $U$  in (27) is real and unitary, and  $\Lambda$  in (27) is diagonal, with diagonal entries that according to (20) are distinct. Furthermore, we can apply rapidly the remaining matrices  $S$  and  $S^{-1}$  in (30) and (31), since  $S$  and  $S^{-1}$  in (30) and (31) are diagonal, once we use (37) and the algorithms from Subsection 3.1 to compute the entries of  $S$  and  $S^{-1}$ .

**Remark 37** Using the algorithms described in the present subsection, we can construct fast algorithms both for computing the coefficients in linear combinations of spherical harmonics, given the values of these linear combinations at certain points, and, vice versa, for evaluating such linear combinations at those points, given the coefficients in the linear combinations. We can handle spherical harmonics by constructing fast algorithms for what are known as associated Legendre functions; see, for example, [13]. For any nonnegative integers  $l$  and  $m$ , the normalized associated Legendre function of order  $m$  and degree  $l$  (often denoted by  $\bar{P}_l^m$ ) is equal to the function  $q_{l-m}$  defined in (22) for the orthonormal polynomials on  $[-1, 1]$  for the weight  $w$  defined by

$$w(x) = (1 - x)^m (1 + x)^m. \quad (83)$$

Thus, we could utilize the algorithms discussed in the present subsection exactly as described, with (40) guaranteeing that the condition number of  $S$  is never too large for practical ranges of  $l$  and  $m$ . However, we would want to take advantage of the symmetries of associated Legendre functions, by handling the even and odd functions separately, using the recurrence relation associated with  $x^2 \bar{P}_l^m(x)$  instead of the recurrence relation (15), which is associated with  $x \bar{P}_l^m(x)$ . We might also want to use the Christoffel-Darboux identity to compute interpolations to and from values at the zeros of various polynomials, as originated in [7] and [18], and

subsequently optimized and extended (see, for example, [8] and Remarks 11 and 15 in [16]).

### 3.4 Evaluations of linear combinations of Bessel functions

We can apply rapidly the matrix  $U^T$  in (67) using an algorithm as in the third item in Subsection 2.1, since  $T$  defined in (45) is tridiagonal, real, and self-adjoint, and (according to Corollary 23) has  $n$  distinct eigenvalues, and hence  $U$  in (67) can be chosen to be real and unitary. Furthermore, we can apply rapidly the remaining matrix  $S$  in (67), since  $S$  in (67) is diagonal, once we use (69), an algorithm as in the third section in Subsection 2.1, and the algorithm from Subsection 3.2 to compute the entries of  $S$ .

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